Calc BC Polar WS2

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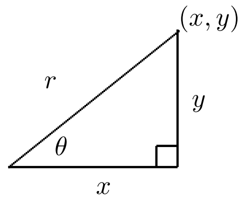


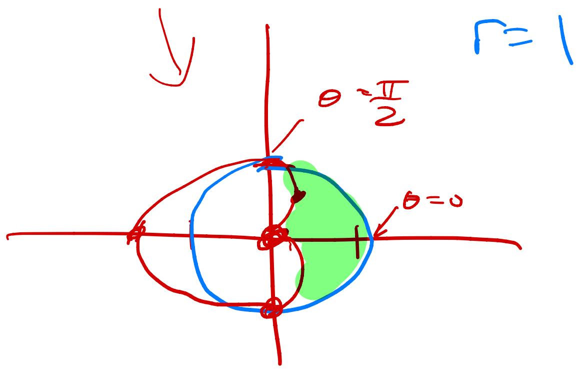
Polar Functions

Cartesian/Polar Equations (x, y) vs (r, θ)
$x^2 + y^2 = r^2$
$x = r\cos\theta$
$y = r\sin\theta$
$\theta = \arctan\dfrac{y}{x}$
Area$= \dfrac{1}{2}r^2\theta$

The following formulas will be used on the AP Calculus exam. You need to know them.

The first derivative (the change in y with respect to x) so if $r = f(\theta)$, using the product rule, the complete polar form is	$\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{(r\sin\theta)'}{(r\cos\theta)'}$ $= \dfrac{f(\theta)\cos\theta + f'(\theta)\sin\theta}{-f(\theta)\sin\theta + f'(\theta)\cos\theta}$
The Area **INSIDE** a polar curve is given by	$\dfrac{1}{2}\displaystyle\int_\alpha^\beta r^2\ d\theta$
To find the area **BETWEEN** polar curves, sketch a wedge-shaped $d\theta$ and find the angle coordinates of the points of intersection. The area will be	$\dfrac{1}{2}\displaystyle\int_\alpha^\beta R^2 - r^2\ d\theta$ or $\dfrac{1}{2}\displaystyle\int_\alpha^\beta (OR)^2 - (IR)^2\ d\theta$
The length along the arc of a polar curve is given by	$\displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$

1. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$



θ	r
0	0
π/2	1
π	1−−1=2
π/4	1−√2/2

R(OR) r (IR)

$$2\cdot\frac{1}{2}\int_0^{\pi/2} 1^2 - (1-\cos\theta)^2\, d\theta$$

$$\int_0^{\pi/2} 1 - (1 - 2\cos\theta + \cos^2\theta)\, dt$$

$$2\int_0^{\pi/2}\cos\theta\, d\theta - \int_0^{\pi/2}\cos^2\theta\, d\theta$$

$$2\sin\theta\Big|_0^{\pi/2} - \left[\frac{1}{2}\int_0^{\pi/2}\cos 2\theta\, d\theta + \int_0^{\pi/2}\frac{1}{2}\, d\theta\right]$$

$u = 2\theta$, $du = 2\,d\theta$, $\frac{1}{2}du = d\theta$, $u(0)=0$, $u(\frac{\pi}{2}) = \pi$

$$2 - \frac{1}{2}\cdot\frac{1}{2}\int_0^{\pi/2}\cos u\, du - \frac{1}{2}\theta\Big|_0^{\pi/2}$$

$$2 - \frac{1}{4}\left[\sin u\Big|_0^{\pi}\right] - \frac{\pi}{4}$$

$$2 - \frac{\pi}{4}$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta + 1 - \cos^2\theta = \cos^2\theta$

$\frac{\cos 2\theta + 1}{2} = \frac{2\cos^2\theta}{2}$

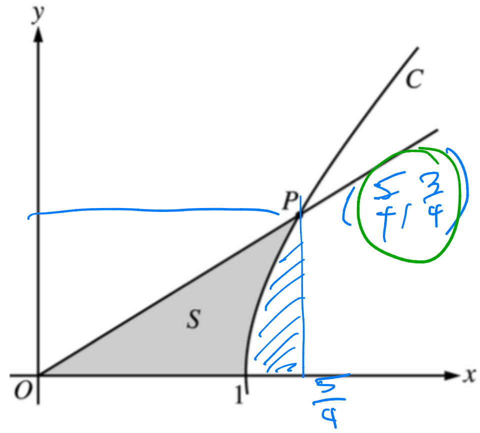
$\frac{1}{2}\cos 2\theta + \frac{1}{2}$

2. 2003 BC Exam Question 3 (calculator allowed)

$x^2 = 1 + y^2$

$2x \cdot \frac{dx}{dy} = 0 + 2y \frac{dy}{dy}$

$\frac{dx}{dy} = \frac{2y}{2x}\Big|_{(5/4,\,3/4)} = \frac{3}{5}$



$\sqrt{x^2 - 1} = \sqrt{y^2}$

$\operatorname{atan}\left(\frac{3}{5}\right)$

The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P. $\left(\frac{5}{4}, \frac{3}{4}\right)$

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

$\frac{5}{4} \cdot \frac{1}{2} \cdot \frac{3}{4} - \int_1^{5/4} \sqrt{x^2 - 1}\, dx$

Area $S = 0.3465732929$

(c)

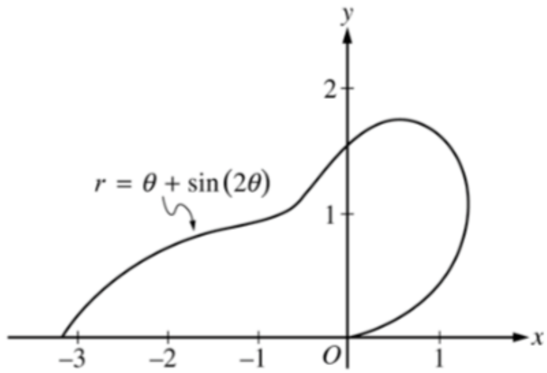
$(r\cos\theta)^2 - (r\sin\theta)^2 = 1$

$r^2(\cos^2\theta - \sin^2\theta) = 1$

$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$

$\frac{1}{2}\int_0^{\arctan 3/5} r^2\, d\theta$

3. 2005 BC Exam Question 2 (calculator allowed)



The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

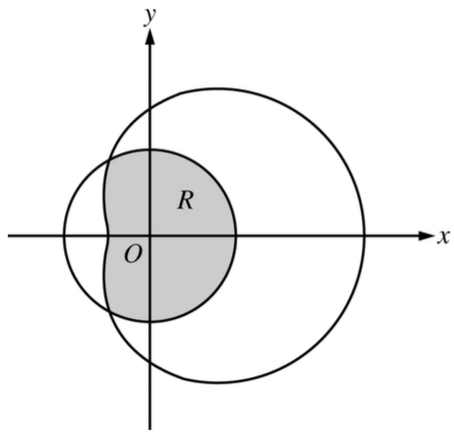
(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

4. 2007 BC Exam Question 3. (calculator allowed).



The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.